

Which Unbounded Protocol for Envy Free Cake Cutting is Better?

Abstract

There are three protocols for envy free cake division for n people. All of them take an unbounded number of cuts. We quantify this unbounded number with ordinals and hence can say, of the three unbounded algorithms, which one is better.

1 Introduction

How do n people split a cake fairly? This is a well studied question, using different definitions of fairness. See for example the books of Robertson&Webb [8] and Brams&Taylor [1]. What makes the topic interesting is that the people may have different tastes. Alice likes the left part that has chocolate, where as Bob likes the right part that has kale. Formally they all have valuations on the cake. A valuation is a function with domain well behaved subsets of the cake and co-domain $[0, 1]$. The entire cake maps to 1 and the valuation is additive.

What is fair?

Def 1.1 There are n players A_1, \dots, A_n .

1. A division is *proportional* if everyone gets a piece of cake that they value as $\geq \frac{1}{n}$.
2. A division is *envy free* if everyone gets a piece of cake that they think is the best or tied for the best.

3. A *discrete protocol* is a procedure that involves only discrete steps and ends with a division of the cake. This is in contrast to moving knife protocols [2]. We will use the term *n-player c-cuts proportional protocol* to mean *discrete protocol that results in a proportional division for n players and uses at most c cuts*. Similar for *envy free protocol*.

There is a 2-player 1-cut proportional protocol: Alice cuts, Bob Choose. Even and Paz [5] proved that, for all n , there is an n -player $O(n \log n)$ proportional protocol. Edmonds and Pruhs [4] proved that $\Omega(n \log n)$ cuts is optimal.

Selfridge and Conway independently obtained a 3-player 5-cut envy free protocol in the early 1960's (unpublished, though in the books on cake cutting cited above). It was an open problem for many years to find a 4-player envy free protocol until, in 1995, Brams and Taylor [3] showed that, for $n \geq 4$ there is an n player envy free protocol. The protocol uses an unbounded number of cuts. For any particular players it will use a finite number of cuts; however, that number depends on the players valuations.

Two more unbounded cuts envy free protocols have been discovered, due to Robertson and Webb [7] and Pikhurto [6]. Of the three known protocols, which is better? We need a way to compare unbounded protocols.

Def 1.2 Let ζ be an ordinal. A protocol takes ζ cuts if (1) the protocol starts with ζ in a counter, (2) every time a cut is made an outside observer who knows only what all of the players know at the time, and wants the protocol to succeed, decreases the counter, and (3) at the end of the protocol the counter has a number ≥ 0 . Note that if (say) ω is in the counter and a cut is made then ω will be replaced by some natural number, though it might be large. The outside observer will know, when the time comes to decrease the counter, how many cuts are needed to finish the protocol.

We will sketch variants of the three envy free protocols and analyze them in terms of how many cuts they take, both in the worst case and in the average case. The variants use the essential ideas but try to optimize the number of cuts.

In describing protocols we use the convention (from [3]) that what a player has to do is described and what the player is advised to do is written in parenthesis. By *should do* we mean that if a player A does not follow the advice then A might end up with less than $\frac{1}{n}$. We do not prove these assertions.

Our results are as follows:

1. **The Brams and Taylor Protocol:** Let $n \in \mathbb{N}$ and $L = LCM(2, \dots, n)$.

- (a) The protocol uses $\left\lceil \frac{n^2-2n+2}{2} \right\rceil \omega + L - 1$ -cuts in the worst case.
- (b) The protocol uses $(n + o(1))\omega + L - 1$ cuts in the average case (defined suitably).

2. **The Robertson and Webb Protocol:**

- (a) The protocol uses $(2n - 3)\omega$ -cuts in the worst case.
- (b) The protocol uses $(2n - O(1))\omega$ cuts in the average case (defined suitably).

3. **The Pikhurto Protocol** is similar to the Robertson and Webb protocol.

2 The Brams-Taylor Protocol

Def 2.1 Let C be a cake to be split and let A_1, \dots, A_n be the ones who will split it. A_i has an *advantage over* A_j if A_i does not care how much of C A_j gets.

Lemma 2.2 *There is an n -player, ω -cuts protocol which will do the following. The players are A_1, \dots, A_n .*

1. *The input is three pieces P, Q, R such that there are two players A_i, A_j where A_i think P and Q are the same size, but A_j thinks P and Q are different sizes.*
2. *At the end of the protocol all but a (small) piece T of P, Q, R are divided amongst A_1, \dots, A_n .*
3. *The division of $P \cup Q \cup R - T$ to A_1, \dots, A_n is envy free.*

4. A_i and A_j each have an advantage over each other with regard to dividing T .

Def 2.3 We call the protocol from Lemma 2.2 *the adv $(A_1, \dots, A_n; A_i, A_j; P, Q, R)$ -protocol*.

Lemma 2.4 *Let G be a graph on n vertices and e edges. If $e \geq \left\lceil \frac{n(n-2)}{2} + 1 \right\rceil = \left\lceil \frac{n^2-2n+2}{2} \right\rceil$ then G must have a vertex of degree $n - 1$.*

Proof: We prove the contrapositive. If every vertex is of degree $\leq n - 2$ then

$$2e = \sum_{v \in V} d_v \leq n(n - 2)$$

$$\text{so } e \leq \frac{n(n-2)}{2}. \quad \blacksquare$$

Theorem 2.5 *Let $n \in \mathbb{N}$ and $L = \text{LCM}(2, \dots, n)$. There is an n -person, envy free protocol that has the following properties.*

1. *The protocol uses $\left\lceil \frac{n^2-2n+2}{2} \right\rceil \omega + L - 1$ -cuts in the worst case.*
2. *The protocol uses $(n + o(1))\omega + L - 1$ cuts in the average case (defined suitably).*

Proof:

We give a protocol that has as input a cake C and a graph G on n vertices. If (A, B) is an edge in G then A and B have an advantage over each other with regard to how C is split. We will denote the edges of G by E .

The protocol is denoted $EFBT(C, G)$ (Envy Free Brams-Taylor). It may call itself with a much smaller cake and a slightly bigger graph. The players are A_1, \dots, A_n .

PROTOCOL $EFBT(C, G)$.

1. If there is a vertex A of degree $n - 1$ in G then nobody else cares if A gets more cake than they do. So give all of the cake to A and the protocol ends. Otherwise proceed.

2. A_1 divides the cake into $L = LCM(2, 3, \dots, n)$ pieces. (Equally.) She uses $L - 1$ cuts.
3. Everyone writes down either EQ or NEQ . A_1 has to write down EQ . (A_i writes EQ if A_i thinks that all of the pieces are equal, NEQ if A_i thinks that two of the pieces are not equal.)
4. What everyone wrote is revealed. Partition the people into two groups EQ and NEQ based on what they wrote.
5. Form the bipartite graph $H = (EQ, NEQ, E \cap (EQ \times NEQ))$. Note that since $A_1 \in EQ$, $EQ \neq \emptyset$.
6. **Case 1:** H is a complete bipartite graph (this includes the case where $NEQ = \emptyset$). Let k be the number of people in EQ . Nobody in NEQ cares what anyone in EQ gets. The people in EQ think all of the pieces are equal. Each person in EQ gets L/k pieces. Note that k divides L by the definition of L .

Case 2: H is not the complete bipartite graph. Let (i, j) be the least pair lexicographically such that (A_i, A_j) is not an edge. This is not arbitrary: we would like to use (A_1, A_j) if we can. Let P, Q be such that A_i thinks $P = Q$ but A_j thinks $P \neq Q$. Let $R = C - (P \cup Q)$. The protocol $adv(A_1, \dots, A_n; A_i, A_j; P, Q, R)$ is run. This takes ω cuts. Let C' be the cake that is left over. Call $EFBT(C', G \cup \{i, j\})$.

END OF PROTOCOL $EFBT$

To envy free divide a cake among n people you would call $EFBT(C, \emptyset)$. Once G has a vertex of degree $n - 1$ the protocol will stop. Each iteration adds a pair. By Lemma 2.4 the number of iterations is bounded by $\left\lceil \frac{n^2 - 2n + 2}{2} \right\rceil$. Hence the number of cuts is bounded by $\left\lceil \frac{n^2 - 2n + 2}{2} \right\rceil \omega + L - 1$.

What happens in the average case? This needs to be defined. We assume that the partitioning of A_2, \dots, A_n into EQ and NEQ is random. Given this, we show that the expected number of iterations before A_1 has degree $n - 1$ is $n + o(1)$.

Let $E(L)$ be the expected number of iterations before A_1 has degree L or the protocol terminates. Clearly $E(1) = 1$. If A_1 has degree $L - 1$ then the probability that in the next iteration A_1 will gain a degree or $NEQ = \emptyset$ (so the protocol terminates) is $1 - \frac{2^{n-(L-1)}}{2^{n-1}} = 1 - (0.5)^{L+2}$. Hence $E(L) = E(L - 1) + \frac{1}{1 - (0.5)^{L+2}}$; therefore,

$$E(n) = 1 + \sum_{i=2}^n \frac{1}{1 - (0.5)^{L+1}} \sim \frac{\ln(2^{n+1}) - 1}{\ln(2)} = n + o(1).$$

Hence the average case is $(n + o(1))\omega + L - 1$ cuts. ■

Note that the protocol from Theorem 2.5 yields a 4-person 5ω -cuts envy free protocol.

3 Robertson and Webb Protocol

In the definitions below we assume that the cake is normalized to have value 1 for everyone. When we use these definitions we may apply them to a piece of cake that they view differently. We leave it to the reader to make the needed modifications.

Def 3.1 Let $n, p \in \mathbb{N}$ and $0 \leq \epsilon < 1$. A *near-exact* (n, p, ϵ) *protocol* is one that n people participate in, and at the end there exists p pieces of cake such that everyone thinks that every pieces is within ϵ of $\frac{1}{p}$. A *near-exact-** (n, p, ϵ) *protocol* is a near exact (n, p, ϵ) -protocol where one of the players (always A_1) thinks all of the pieces are exactly $\frac{1}{p}$. Note that for near-exact and near-exact-* protocols we do not give cake to anyone.

The following lemma was first proven by Robertson and Webb [7]; however, Pikhurto [6] later had an especially nice proof.

Lemma 3.2 *If $n, p \in \mathbb{N}$ and $0 \leq \epsilon < 1$ then there exists a near exact-* (n, p, ϵ) protocol. The number of cuts is a function of n, p and ϵ .*

Def 3.3 Let $n \in \mathbb{N}$, $0 < f_1, f_2 < 1$, such that $f_1 + f_2 = 1$, $0 \leq \epsilon < 1$. An *unfair near exact* (n, f_1, f_2, ϵ) *protocol* is one that n people participate in, and at the end there exists 2 pieces of cake such that everyone thinks that the first piece is within ϵ of f_1 and the second piece is within ϵ of f_2 . (We will not need the $*$ -version.)

Lemma 3.4 For all $n \in \mathbb{N}$, $0 < f_1, f_2 < 1$, such that $f_1 + f_2 = 1$, $0 \leq \epsilon < 1$ there exists an *unfair* (n, f_1, f_2, ϵ) *protocol*. The number of cuts depends on a, f_1, f_2 , and ϵ .

Def 3.5 Let A_1, \dots, A_n be the people. A piece of cake P is *controversial* if there exists a nontrivial partition of the people into sets S_1 and S_2 , and two numbers $\alpha > \beta$ such that

- Everyone in S_1 thinks that P is worth α .
- Everyone in S_2 thinks that P is worth $\leq \beta$.

Def 3.6 For all $n, m \in \mathbb{N}$ and $\epsilon > 0$ a *controversial* (n, m, δ) -*protocol* is a protocol for $n + m$ people $A_1, \dots, A_n; B_1, \dots, B_m$ that starts with a piece P that is controversial for A_1, \dots, A_n (we do not know what the B_i 's think of P), and ends with a piece P' such that that (1) P' is controversial for A_1, \dots, A_n (though perhaps with a different partition than the controversy of P), and (2) everyone (including the B_i s) thinks P' is worth $\leq \delta$.

Lemma 3.7 For all $n, m \in \mathbb{N}$ and $\epsilon > 0$ there exists an *controversial* (n, m, ϵ) -*protocol*. The number of cuts depends on n, m , and ϵ .

Theorem 3.8 Let $m, n \in \mathbb{N}$ and $\epsilon > 0$. There is a protocol for $n+m$ people $A_1, \dots, A_n; B_1, \dots, B_m$ that divides a cake C into n pieces, each piece going to one of the A -people, (The B -people get nothing!) such that the following happens.

1. The division is envy free for A_1, \dots, A_n .
2. $A_1, \dots, A_n, B_1, \dots, B_m$ all think that every piece is within ϵ of $\frac{1}{n}$.

The number of cuts is as follows.

1. The protocol uses $(2n - 3)\omega$ -cuts in the worst case.
2. The protocol uses $(2n - O(1))\omega$ cuts in the average case (defined suitably).

Proof:

We denote the protocol *EFRW* (Envy Free Robertson-Webb). It may call itself twice with some of the A_i 's shifted to the B -side, and with part of the cake.

PROTOCOL *EFRW*($A_1, \dots, A_n; B_1, \dots, B_m; C; \epsilon$)

1. If $n = 1$ then give A_1 the entire cake and the protocol is done.
2. If $n = 2$ then $A_1, A_2, B_1, \dots, B_m$ run a near-exact $(m + 2, 2, \epsilon)$ -protocol on the cake to produce two pieces that A_1 thinks are identical and everyone else thinks are within ϵ of $\frac{1}{2}$. This takes ω cuts. A_2 picks and keeps one of the pieces, A_1 keeps the other. The protocol is done.
3. (It must be that $n \geq 3$.) $A_1, \dots, A_n; B_1, \dots, B_m$ run a near-exact* $(n + m, n, \epsilon)$ -protocol. This takes ω cuts. If A_2, \dots, A_n agree with A_1 that these pieces are all of size $\frac{1}{n}$, then these pieces are given out (it does not matter how) and the protocol is done. Else goto the next step.
4. There is a piece P that is controversial for A_1, \dots, A_n . Let δ be a parameter to be picked later (it will depend on ϵ, n, m). $A_1, \dots, A_n; B_1, \dots, B_m$ run a controversial (n, m, δ) -protocol. This step takes ω cuts.
5. There is a piece P , numbers $\beta < \alpha \leq \delta$ and (after renumbering) $1 \leq i \leq n - 1$ such that

- A_1, \dots, A_i all think P is worth α
 - A_{i+1}, \dots, A_n all think P is worth $\leq \beta$.
 - $A_1, \dots, A_n, B_1, \dots, B_m$ all think P is worth $\leq \delta$.
6. Let $Q = C - P$. $A_1, \dots, A_n; B_1, \dots, B_m$ run an *unfair* $(n + m, f_1, f_2, \epsilon)$ -protocol to split Q into Q_1 and Q_2 with f_1, f_2 picked such that all think Q_1 is just a shade less than i/n of Q and Q_2 is just a shade more than $(n - i)/n$ of Q . That shade is a function of n, m, ϵ and $a - b$.
7. Run $EFRW(A_1, \dots, A_i; A_{i+1}, \dots, A_n, B_1, \dots, B_m; Q_1 \cup P; \epsilon')$ (note that A_{i+1}, \dots, A_n are now on the B -side) where ϵ' will be discussed later. $Q_1 \cup P$ is divided into i pieces and given to A_1, \dots, A_i in an envy free manner, while $A_{i+1}, \dots, A_n, B_1, \dots, B_m$ think each piece is within ϵ' of $\frac{1}{i}$ of $Q_1 \cup P$.
8. Run $EFRW(A_{i+1}, \dots, A_n; A_1, \dots, A_i, B_1, \dots, B_m; Q_2; \epsilon')$ (note that A_1, \dots, A_i are now on the B -side). Q_2 is divided into $n - i$ pieces and given to A_{i+1}, \dots, A_n in an envy free manner while $A_1, \dots, A_i, B_1, \dots, B_m$ think each piece is within ϵ' of $\frac{1}{n-i}$ of Q_2 .

We pick that *shade less than i/n* carefully: close enough to i/n so that A_1, \dots, A_i think that getting $Q_1 \cup P$ is worth getting a shade less than i/n , but big enough so that A_{i+1}, \dots, A_n thinks that getting that shade is worth more than P . Such a shade exists since A_1, \dots, A_i value P more than A_{i+1}, \dots, A_n . We pick ϵ' so small that (1) A_1, \dots, A_i do not mind that A_{i+1}, \dots, A_n may get ϵ' more than $\frac{n-i}{n}$ of Q_2 , and (2) A_{i+1}, \dots, A_n do not mind that A_1, \dots, A_i may get ϵ' more than $\frac{i}{n}$ of $Q_1 \cup P$.

What about the B_i s? The parameter δ and ϵ' are picked small enough so that at the end the B_i s see A_1, \dots, A_n getting within ϵ of $\frac{1}{n}$.

Let $T(n; m)$ be the number of cuts this protocol takes.

$$(\forall m \geq 0)[T(1; m) = 0]$$

$$(\forall m \geq 0)[T(2; m) = \omega]$$

If $n \geq 3$ and $m \geq 1$ then the protocol will take 2ω cuts and then recurse. Hence

$$T(n; m) \leq 2\omega + \max_{1 \leq i \leq n-1} (T(i; n + m - i) + T(n - i; m + i))$$

One can easily show that $(\forall n \geq 1)(\forall m \geq 0)[T(n; m) \leq (2n - 3)\omega]$. In particular $T(n; 0) \leq (2n - 3)\omega$. Hence when used for n -player envy free cake cutting, this protocol takes $(2n - 3)\omega$ cuts in the worst case.

We can study the average case by assuming that the players partitioning is random. This leads to an average case of $(2n - O(1))\omega$.

■

Note that the protocol from Theorem 3.8, yields a 4-person 5ω -cuts envy free protocol. Hence, for the case of $n = 4$, it uses (essentially) the same number of cuts as the protocol from Theorem 2.5.

4 Pikhurto's Protocol

For our purposes Pikhurto's protocol is similar to the Robertson-Webb protocol so we discuss it briefly and informally.

In the Robertson-Webb protocol the A -players are partitioned into *two* groups: those who think P is size α and those who think P is of size $\leq \beta$. In Pikhurto's protocol the A -players are partitioned into many groups and within a group the opinion of P is the same. Then the protocol calls itself on each group.

Let $T(n; m)$ be the number of cuts this protocol takes.

$$(\forall m \geq 0)[T(1; m) = 0]$$

$$(\forall m \geq 0)[T(2; m) = \omega]$$

If $n \geq 3$ and $m \geq 1$ then the protocol will take 2ω cuts and then recurse on each group. Hence

$$T(n; m) \leq 2\omega + \max_{\{i_1, \dots, i_k : i_1 + \dots + i_k = n\}} T(i_1; n + m - i_1) + \dots + T(i_k; n + m - i_k)$$

One can easily show that $T(n; m) \leq (2n - 3)\omega$ and that if the partition of the players is random then the average case is $(2n - O(1))\omega$.

5 Open Problems

Is there an n -player $\omega + O(1)$ -cut envy free protocol? Can the results for small values of n be improved from what we have here? Sam Zbarsky has obtained (unpublished) a protocol for $n = 4$ that takes only $2\omega + O(1)$ cuts, in contrast to what we obtained which was $5\omega + O(1)$ cuts. His approach is rather complicated and does not seem to generalize; however, it is a proof-of-concept that special case algorithms may do better than those presented in our paper.

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